Materials discussed on 15/11

Using Schwarz Lemma, we show that for any $f : \mathbb{D} \to \mathbb{D}$,

$$\left| \frac{f(z) - f(w)}{1 - \bar{f}(w) f(z)} \right| \leq \left| \frac{z - w}{1 - \bar{w} z} \right|$$

for $z, w \in \mathbb{D}$. And equality holds if and only if $f \in Aut(\mathbb{D})$.

This motivates us to define pseudo-hyperbolic metric to be

$$\tilde{d}(z, w) = \left| \frac{z - w}{1 - \bar{w} z} \right|.$$

And thus the Schwarz Pick lemma:

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D}.$$

In the infinitesimal level, we define metric to be

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}.$$

We sometimes call this the first fundamental form. And therefore we can define distance to be

$$d(z, w) = \inf_{\gamma : z \to w} L(\gamma)$$

where the inf is taken over all curve from z to w, parameterized by $[0, 1]$ and

$$L(\gamma) = \int_0^1 ||\gamma'|| dt.$$

Here we use the inner product defined above. More Precisely, if $\gamma = (\gamma_1, \gamma_2) \in \mathbb{D}$, then

$$||\gamma'|| = \frac{1}{(1 - |\gamma|^2)} \left[(\gamma_1')^2 + (\gamma_2')^2 \right]^{1/2}.$$

In other word, the metric is conformal to Euclidean with conformal factor $(1 - |z|^2)^{-2}$. In fact, $(\mathbb{D}, d)$ is a length space. In differential geometry, $(\mathbb{D}, ds^2)$ is a spaceform with sectional curvature -1 which is very important.

Question: Any relations between $Aut(\mathbb{D})$ and $Iso(\mathbb{D})$?

Proposition 0.1. *$Aut(\mathbb{D}) \subset Iso(\mathbb{D})$*

Proof. Let $\gamma : z \to w$ parameterized by $[0, 1]$, then $f \circ \gamma : f(z) \to f(w)$.

$$\partial_t (f \circ \gamma(t)) = \frac{\partial f}{\partial z} \cdot dz(\gamma'(t)).$$

Since $dz = dx + idy$ and

$$dx(\gamma') = \gamma_1' \quad \text{and} \quad dy(\gamma') = \gamma_2'$$

if we write $\gamma = (\gamma_1, \gamma_2) \in \mathbb{D}$. Hence,

$$|(f \circ \gamma)'| = |f'|\sqrt{\gamma_1'^2 + \gamma_2'^2} = |f'(\gamma(t))||\gamma'(t)|$$

Result follows from Schwarz Pick lemma and taking inf over all curve. Replace f by f^{-1} to conclude that it is isometry. □

Proposition 0.2. *If $f \in Iso(\mathbb{D})$, then either $f \in Aut(\mathbb{D})$ or $\bar{f} \in Aut(\mathbb{D})$.*

Proof. By fractional transformation, we have for all $z, w \in \mathbb{D}$,

$$d(z,w) = d(0, |\frac{z-w}{1-\bar{w}z}|).$$

Denote $s = |\frac{z-w}{1-\bar{w}z}| > 0$. Let $\sigma = (x(t), y(t))$ be any curve from 0 to s.

$$\int_0^1 \frac{\sqrt{x'^2 + y'^2}}{1-(x^2+y^2)} dt \geq \int_0^1 \frac{x'(t)}{1-x(t)^2} dt = \frac{1}{2}\log\frac{1+s}{1-s}.$$

From this, it can also been seen that $\gamma(t) = (0, st)$ is the curve capturing the distance. It becomes the geodesic if it is normalised to have unit length.

$$d(z,w) = 2\tanh^{-1}\left|\frac{z-w}{1-\bar{w}z}\right|.$$

By fractional transformation, we may assume $f(0) = 0$ and $f(1/2) = 1/2$. Since f preserves the distance, it maps $B(r)$ to $B(r)$. Therefore, $f(i/2) \in B(1/2)$. Consider the equation

$$d\left(\frac{1}{2}, \frac{i}{2}\right) = d\left(\frac{1}{2}, f(\frac{i}{2})\right) = 2\tanh^{-1}\left|\frac{2^{-1} - 2^{-1}e^{i\theta}}{1 - 4^{-1}e^{-i\theta}}\right|.$$

It can be checked that there are at most two solutions on $[0, 2\pi)$. So $f(i/2) = i/2$ or $-i/2$. By taking conjugation, we may assume $f(i/2) = i/2$.

On the other hand, since fractional transformation preserves distance and it maps circles to circles (here we treat straight line to be generalised circle as well). Hence, for any $p \in \mathbb{D}$, $B(p, r)$ is still a circle on $\mathbb{D}$. For any $z \in \mathbb{D}$,

$$d(f(z), 1/2),\ d(f(z), i/2),\ d(f(z), 0)$$

are all known due to the isometry. As three circles can at most meet at a point, $f(z)$ is uniquely determined. Therefore, $f(z) = z$ by uniqueness which is a automorphism.

□